

Orbit equivalence types of circle diffeomorphisms with a Liouville rotation number

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ABSTRACT. This paper is concerned about the orbit equivalence types of C^∞ diffeomorphisms of S^1 seen as nonsingular automorphisms of (S^1, m) , where m is the Lebesgue measure. Given any Liouville number α , it is shown that each of the subspace formed by type II_1 , II_∞ , III_λ ($\lambda > 1$), III_∞ and III_0 diffeomorphisms are C^∞ -dense in the space of the orientation preserving C^∞ diffeomorphisms with rotation number α .

1. Introduction

Let (X, \mathcal{B}, μ) and (X', \mathcal{B}', μ') be Lebesgue measure spaces. A map

$$f : (X, \mathcal{B}, \mu) \rightarrow (X', \mathcal{B}', \mu')$$

is called a *nonsingular isomorphism* (a nonsingular *automorphism* when $(X', \mathcal{B}', \mu') = (X, \mathcal{B}, \mu)$), if f is a bimeasurable bijection from a conull set of X onto a conull set of X' and if $f_*\mu$ is equivalent to μ' .

Two nonsingular automorphisms

$$f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu) \quad \text{and} \quad f' : (X', \mathcal{B}', \mu') \rightarrow (X', \mathcal{B}', \mu')$$

are called *orbit equivalent* if there is a measurable isomorphism

$$h : (X, \mathcal{B}, \mu) \rightarrow (X', \mathcal{B}', \mu')$$

which sends the f -orbit of almost any point to an f' -orbit (regardless of the orders of the orbits).

A nonsingular automorphism

$$f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$$

is called *ergodic* if any f -invariant measurable subset of X is either null or conull.

An ergodic nonsingular automorphism f is said to be of type II if there is a f -invariant σ -finite measure ν which is equivalent to μ . More specifically f is called of type II_1 (resp. of type II_∞) if the f -invariant measure ν is finite (resp. infinite). It is known that they are mutually exclusive.

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An ergodic nonsingular automorphism f is said to be of type III if it is not of type II. Type III nonsingular automorphism f is further classified according to the *ratio set* $r(f) \subset [0, \infty)$, defined as follows. A number $\alpha \in [0, \infty)$ belongs to $r(f)$ if and only if for any $A \in \mathcal{B}$ with $\mu(A) > 0$ and $\epsilon > 0$, there is $B \in \mathcal{B}$ and $i \in \mathbb{Z}$ such that $\mu(B) > 0$, $B \subset A$, $f^i(B) \subset A$, and for any $x \in B$, the Radon-Nikodym derivative satisfies:

$$\frac{d(f^{-i})_*\mu}{d\mu}(x) \in (\alpha - \epsilon, \alpha + \epsilon).$$

The definition of the ratio set $r(f)$ does not depend on the choice of a measure μ from its equivalence class $[\mu]$. Moreover it is an invariant of an orbit equivalence class. The ratio set $r(f)$ is a closed subset of $[0, \infty)$, and $r(f) \cap (0, \infty)$ is a multiplicative subgroup of $(0, \infty)$. It is well known, easy to show, that f is of type II if and only if $r(f) = \{1\}$. For f of type III, we have the following classification.

- f is of type III_λ for some $\lambda > 1$ if $r(f) = \lambda^\mathbb{Z} \cup \{0\}$.
- f is of type III_∞ if $r(f) = [0, \infty)$.
- f is of type III_0 if $r(f) = \{0, 1\}$.

It is known ([**Kr**]) that the set of ergodic nonsingular transformations of type II_1 , II_∞ , III_λ , III_∞ each consists of one orbit equivalence class. But the set of the ergodic nonsingular transformations of type III_0 consists of various classes.

Let F be the group of the orientation preserving C^∞ diffeomorphisms of S^1 . For $\alpha \in \mathbb{R}/\mathbb{Z}$, denote by F_α the subset of F consisting of those diffeomorphisms f whose rotation number $\rho(f)$ is α . If α is irrational, then any element $f \in F_\alpha$ is ergodic with respect to the Lebesgue measure m ([**H**], p.86, [**KH**] 12.7).

In [**Ka**], Y. Katznelson has shown that the diffeomorphisms of any type raised above are C^∞ dense in the union of F_α for irrational α .

In this paper we focus on a single subset F_α . For any $f \in F_\alpha$, α irrational, there is a unique homeomorphism, denoted by H_f , such that $H_f(0) = 0$ and $f = H_f R_\alpha H_f^{-1}$, where R_α stands for the rotation by α . Thus the unique f invariant measure on S^1 is given by $(H_f)_*m$. This implies that either $(H_f)_*m$ is equivalent to m or else singular to m .

For a non Liouville number α , it is shown ([**Y1**]) that H_f is a C^∞ diffeomorphism for any $f \in F_\alpha$. That is, $(H_f)_*m$ is equivalent to the Lebesgue measure m , and hence any $f \in F_\alpha$ is of type II_1 .

But for a Liouville number α , things are quite different. There are $f \in F_\alpha$ for which the unique f invariant measure $(H_f)_*m$ is singular to m ([**M1**, **M2**]). Such f can never be of type II_1 . Denote by $F_\alpha(\text{T})$ the subset of F_α consisting of diffeomorphisms of type T. The main result of this paper is the following.

THEOREM 1. *For any Liouville number α , each of the subsets $F_\alpha(\text{II}_1)$, $F_\alpha(\text{II}_\infty)$, $F_\alpha(\text{III}_\lambda)$ for any $\lambda > 1$, $F_\alpha(\text{III}_\infty)$ and $F_\alpha(\text{III}_0)$ forms a C^∞ -dense subset of F_α .*

The key fact for the proof is the result in [**Y2**] which states that even for a Liouville number α , the subset of elements $f \in F_\alpha$ such that H_f are C^∞ diffeomorphisms is C^∞ dense in F_α . Since the C^∞ closure of our subset $F_\alpha(\text{T})$ is invariant by the conjugation by an element $f \in F$, it suffices to show the following proposition.

PROPOSITION 1.1. *For any $r \in \mathbb{N}$, there is an element $f \in F_\alpha(\text{T})$ such that $d_r(f, R_\alpha) < 2^{-r}$, where d_r is the C^r distance and $\text{T} = \text{II}_\infty, \text{III}_\lambda, \text{III}_\infty, \text{III}_0$.*

Proposition 1.1 is proved by the method of fast approximation by conjugacy with estimate, developed in [**FS**]. This is a qualitatively refined version of the

method of successive approximations originated by D. Anosov and A. Katok [AK] in the early 70's.

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2. Fast approximation method

We assume throughout that $0 < \alpha < 1$ is a Liouville number, i. e. for any $N \in \mathbb{N}$ and $\epsilon > 0$, there is p/q ($p, q \in \mathbb{N}$, $(p, q) = 1$) such that

$$(2.1) \quad |\alpha - p/q| < \epsilon q^{-N}.$$

To prove Proposition 1.1 for $F_\alpha(\mathbb{T})$, we will actually show the next proposition.

PROPOSITION 2.1. *For any $r \in \mathbb{N}$, there are sequences $\alpha_n = p_n/q_n \in \mathbb{Q} \cap (0, 1)$, $(p_n, q_n) = 1$, and $h_n \in F$ ($n \in \mathbb{N}$) such that the following (i) \sim (v) hold. Define $H_0 = \text{Id}$, $f_0 = R_{\alpha_1}$, and for any $n \in \mathbb{N}$*

$$H_n = h_1 \cdots h_n \quad \text{and} \quad f_n = H_n R_{\alpha_{n+1}} H_n^{-1}.$$

- (i) $\alpha_n \rightarrow \alpha$.
- (ii) R_{α_n} commutes with h_n .
- (iii) H_n converges uniformly to a homomorphism H .
- (iv)

$$|\alpha - \alpha_1| < 2^{-r-1}, \quad \text{and} \quad d_{n+r}(f_{n-1}, f_n) < 2^{-n-r-1}, \quad \forall n \geq 1.$$

- (v) The limit f of $\{f_n\}$ is an element of $F_\alpha(\mathbb{T})$.

Notice that (iv) implies that the limit f of f_n is a C^∞ diffeomorphism such that $d_r(f, R_\alpha) < 2^{-r}$.

Condition (ii) is useful to establish (iv), since then

$$\begin{aligned} f_{n-1} - f_n &= H_n R_{\alpha_n} H_n^{-1} - H_n R_{\alpha_{n+1}} H_n^{-1}, \\ f_{n-1}^{-1} - f_n^{-1} &= H_n R_{-\alpha_n} H_n^{-1} - H_n R_{-\alpha_{n+1}} H_n^{-1}, \end{aligned}$$

and these can be estimated using Lemma 2.3 below.

Next we shall summarize inequalities needed to establish Proposition 2.1. All we need are polynomial type estimates whose degree and coefficients can be arbitrarily large. The inequalities below are sometimes far from being optimal.

For a C^∞ function φ on S^1 , we define as usual the C^r norm $\|\varphi\|_r$ ($0 \leq r < \infty$) by

$$\|\varphi\|_r = \max_{0 \leq i \leq r} \sup_{x \in S^1} |\varphi^{(i)}(x)|.$$

For $f, g \in F$, define

$$\begin{aligned} |||f|||_r &= \max\{\|f - \text{id}\|_r, \|f^{-1} - \text{id}\|_r, 1\}, \\ d_r(f, g) &= \max\{\|f - g\|_r, \|f^{-1} - g^{-1}\|_r\}. \end{aligned}$$

The term $|||f|||_r$ is used to show that f is not so large in the C^r -topology. We have included 1 in its definition because then it becomes possible to reduce inequalities from the Faà di Bruno formula ([H], p.42 or [S]) by virtue of the following;

$$|||f|||_r^i \leq |||f|||_r^r \quad \text{if} \quad i \leq r.$$

On the other hand $d_r(f, g)$ is useful for showing f and g are near in the C^r -topology. We get the following inequality from the Faà di Bruno formula.

Below we denote by $C(r)$ an arbitrary constant which depends only on r .

LEMMA 2.2. *For $f, g \in F$ we have*

$$\begin{aligned} \|fg - g\|_r &\leq C(r) \|f - \text{Id}\|_r \|g\|_r^r, \\ \| \|fg\| \|_r &\leq C(r) \| \|f\| \|_r^r \| \|g\| \|_r^r. \end{aligned}$$

□

The next lemma can be found as Lemma 5.6 of [FS] or as Lemma 3.2 of [S].

LEMMA 2.3. *For $H \in F$ and $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$,*

$$d_r(HR_\alpha H^{-1}, HR_\beta H^{-1}) \leq C(r) \| \|H\| \|_{r+1}^{r+1} |\alpha - \beta|.$$

□

For $Q \in \mathbb{N}$, denote by $\pi_Q : S^1 \rightarrow S^1$ the cyclic Q -fold covering map.

LEMMA 2.4. *Let h be a lift of $\hat{h} \in F$ by π_Q and assume $\text{Fix}(h) \neq \emptyset$. Then we have for any $r \geq 0$*

$$\begin{aligned} \|h - \text{Id}\|_r &= \| \hat{h} - \text{Id} \|_r Q^{r-1}, \\ \| \|h\| \|_r &\leq \| \| \hat{h} \| \|_r Q^{r-1}. \end{aligned}$$

PROOF. Just notice that a lift \tilde{h} of h to \mathbb{R} is the conjugate of a lift $\hat{\tilde{h}}$ of \hat{h} by a homothety by Q , i. e. $\tilde{h}(x) = Q^{-1} \hat{\tilde{h}}(Qx)$. □

First of all we give beforehand a sequence of diffeomorphisms $\hat{h}_n \in F$ such that $\hat{h}_n(0) = 0$. The sequence $\{\hat{h}_i\}$ depends upon the type $T = \text{II}_\infty, \text{III}_\lambda, \text{III}_\infty, \text{III}_0$, and will be constructed concretely in the later sections.

Next we choose a sequence of rationals $\alpha_n = p_n/q_n$ inductively in a way to be explained shortly, and set h_n to be the lift of \hat{h}_n by the cyclic Q_n -fold covering map

$$\pi_{Q_n} : S^1 \rightarrow S^1$$

such that $\text{Fix}(h_n) \neq \emptyset$, where $Q_n = K(n)q_n$ and the integer $K(n)$ are chosen to depend only on $\{\hat{h}_i\}_{i \in \mathbb{N}}$ and q_1, \dots, q_{n-1} . Notice that then condition (ii) of Proposition 2.1 is automatically satisfied.

We always choose the rationals α_n so as to satisfy

$$|\alpha_{n+1} - \alpha| < |\alpha_n - \alpha|, \quad \forall n \in \mathbb{N}.$$

Therefore we have

$$|\alpha_n - \alpha_{n+1}| \leq 2|\alpha - \alpha_n|.$$

In the sequel, we denote any constant which depends only on $r, \{\hat{h}_i\}_{i \in \mathbb{N}}$ and $\alpha_1, \dots, \alpha_{n-1}$ by $C(n, r)$. Thus $C(n, r)$ depends only on the initial data about \hat{h}_i and the previous step of the induction. We also denote any positive integer which depends on $r, \{\hat{h}_i\}_{i \in \mathbb{N}}$ and $\alpha_1, \dots, \alpha_{n-1}$ by $N(n, r)$.

By Lemma 2.4, we have for any $1 \leq i < n$,

$$(2.2) \quad \| \|h_i\| \|_{n+r+1} \leq \| \| \hat{h}_i \| \|_{n+r+1} Q_i^{n+r} = C(n, r),$$

and

$$(2.3) \quad \| \|h_n\| \|_{n+r+1} \leq \| \| \hat{h}_n \| \|_{n+r+1} K(n)^{n+r} q_n^{n+r} = C(n, r) q_n^{N(n, r)}.$$

Of course the two $C(n, r)$'s in (2.2) and (2.3) are different. Now we obtain inductively using Lemma 2.2 that

$$(2.4) \quad \|H_n\|_{n+r+1} \leq C(n, r) q_n^{N(n, r)}.$$

The terms $C(n, r)$ and $N(n, r)$ in (2.4) are computed from (2.2) and (2.3) by applying Lemma 2.2 successively.

By Lemma 2.3 and (2.4),

$$(2.5) \quad d_{n+r}(f_{n-1}, f_n) = d_{n+r}(H_n R_{\alpha_n} H_n^{-1}, H_n R_{\alpha_{n+1}} H_n^{-1})$$

$$(2.6) \quad \begin{aligned} &\leq C(n, r) q_n^{N(n, r)} |\alpha_n - \alpha_{n+1}| \\ &\leq C(n, r) q_n^{N(n, r)} |\alpha - \alpha_n|, \end{aligned}$$

for some other $C(n, r)$ and $N(n, r)$.

In order to obtain (iv) of Proposition 2.1, the rational $\alpha_n = p/q$ have only to satisfy

$$C(n, r) q^{N(n, r)} |\alpha - p/q| < 2^{-n-r-1},$$

that is,

$$(2.7) \quad |\alpha - p/q| < 2^{-n-r-1} C(n, r)^{-1} q^{-N(n, r)}.$$

The terms

$$\epsilon = 2^{-n-r-1} C(n, r)^{-1} \quad \text{and} \quad N = N(n, r)$$

are already determined beforehand or by the previous step of the induction. Since α is Liouville, there exists a rational p/q which satisfies (2.1) for these values of ϵ and N . Setting it p_n/q_n , we establish (iv) for the n -th step of the induction.

In fact there are infinitely many choices of p_n/q_n , which enables us to require more. First of all, since

$$\|H_n - H_{n-1}\|_0 \leq \text{Lip}(H_{n-1}) \|h_n - \text{Id}\|_0 \leq \text{Lip}(H_{n-1}) Q_n^{-1} \leq \text{Lip}(H_{n-1}) q_n^{-1},$$

we obtain that H_n converges uniformly to a continuous map H simply if we choose q_n large enough compared with the Lipschitz constant $\text{Lip}(H_{n-1})$ of H_{n-1} . It is easier to obtain that H_n^{-1} converges. Then H is a homeomorphism, getting condition (iii).

There are other requirements for the choice of α_n , which will appear in the next section.

3. Type III_λ : the construction of \hat{h}_n

In this and the next two sections, we shall prove Proposition 2.1 for $T = \text{III}_\lambda$ and $\lambda > 1$. The diffeomorphism \hat{h}_n ($n \in \mathbb{N}$) is constructed as follows. First consider two affine maps of slope $\lambda^{1/2}$ and $\lambda^{-1/2}$, depicted in Figure 1. They intersect at points $(0, 0)$ and $(a, 1-a)$. Choose a rational number $\delta_n > 0$ such that $\delta_n \downarrow 0$. (The precise condition will be given later.) We join the two affine maps on the intervals $(-\delta_n, \delta_n)$ and $(a - \delta_n, a + \delta_n)$ using bump functions. See Figure 2. Finally the diffeomorphism \hat{h}_n is obtained by adding a positive number so that \hat{h}_n has a fixed point at 0.

Choose four *rational* points $\partial^+ \hat{J}_n^+$, $\partial^- \hat{J}_n^-$, $\partial^+ \hat{J}_n^-$ and $\partial^- \hat{J}_n^+$ each near the points $-\delta_n$, δ_n , $a - \delta_n$ and $a + \delta_n$ as in Figure 2.

Define two intervals by

$$\hat{J}_n^- = [\partial^- \hat{J}_n^-, \partial^+ \hat{J}_n^-], \quad \hat{J}_n^+ = [\partial^- \hat{J}_n^+, \partial^+ \hat{J}_n^+].$$

FIGURE 1.

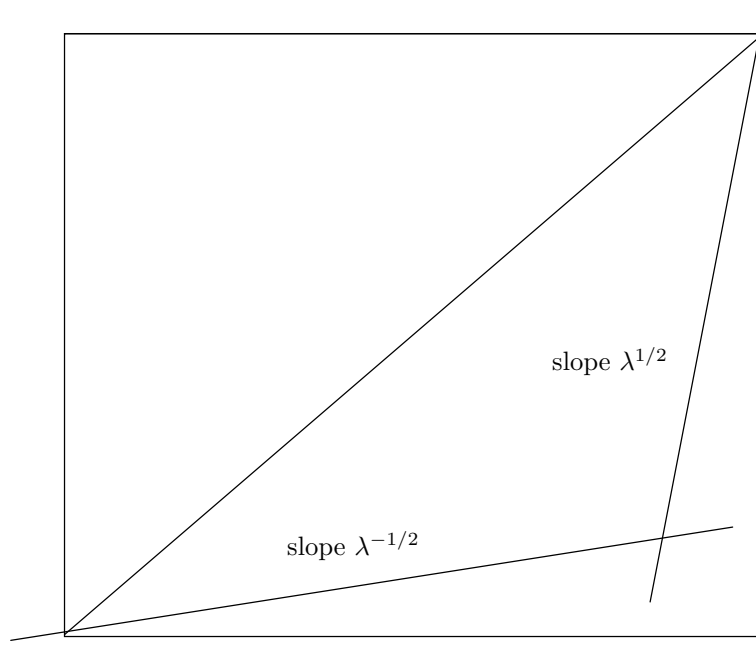
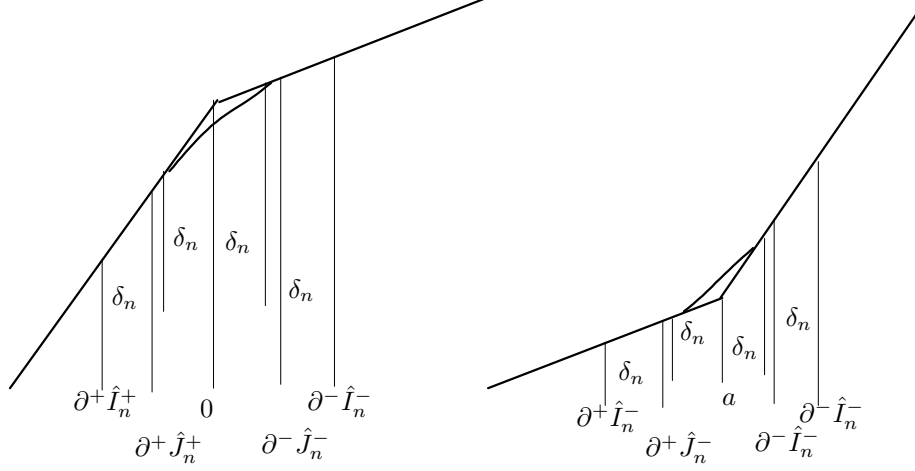


FIGURE 2.



Then the diffeomorphism \hat{h}_n is affine of slope $\lambda^{\pm 1/2}$ on \hat{J}_n^\pm . Next define subintervals \hat{I}_n^\pm of \hat{J}_n^\pm in such a way that the connected components of $\hat{J}_n^\pm - \hat{I}_n^\pm$ have length δ_n . Their boundary points $\partial^\pm I_n^\pm$ is denoted in Figure 2.

Let

$$(3.1) \quad m(\hat{h}_n(\hat{I}_n^- \cup I_n^+)) = 1 - \delta'_n,$$

where m stands for the Lebesgue measure as before. If one choose δ_n tending rapidly to 0, and \hat{J}_n^\pm appropriately, then we have

$$(3.2) \quad \prod_{n=1}^{\infty} (1 - \delta'_n) > 9/10.$$

Let $K'(n)$ be the least common multipliers of the denominators of the eight rational points $\partial^\pm \hat{J}_n^\pm$ and $\partial^\pm \hat{I}_n^\pm$. Define the number $K(n)$ inductively as follows. Let $K(1) = 1$. When we are defining $K(n)$, we already decided q_{n-1} . Set $K(n) = q_{n-1}K'(n-1)$, and notice that this choice of $K(n)$ satisfies the condition of the previous section.

As before, set $Q_n = K(n)q_n$ and h_n to be the lift of \hat{h}_n by the cyclic Q_n -fold covering map π_{Q_n} such that 0 is a fixed point of h_n . Let

$$J_n^\pm = \pi_{Q_n}^{-1}(\hat{J}_n^\pm), \quad I_n^\pm = \pi_{Q_n}^{-1}(\hat{I}_n^\pm).$$

Now J_n^\pm and I_n^\pm each consists of Q_n small intervals. Each of these small intervals, as well as the fixed points of h_n , are left fixed by h_{n+1} . Thus we have inductively

$$(3.3) \quad \text{The boundary points of } J_n^\pm \text{ and } I_n^\pm \text{ are fixed by } h_m \text{ if } m > n.$$

Finally we also assume the following.

$$(3.4) \quad q_{n+1}^{-1} < 2^{-1}Q_n^{-1}\delta_n$$

$$(3.5) \quad |\alpha_n - \alpha| < \delta^2 Q_n^{-2} q_n^{-1}.$$

One can assume (3.4) since q_{n+1} can be chosen arbitrarily large compared with the previous data, and (3.5) since this takes the form (2.1) for the definition of Liouville numbers.

4. Type III_λ : the proof that $r(f) \subset \lambda^\mathbb{Z} \cup \{0\}$

At this point we have already constructed the sequences $\{h_n\}$ and $\{\alpha_n\}$ in Proposition 2.1. Thus for

$$H_n = h_1 \cdots h_n, \quad f_n = H_n R_{\alpha_{n+1}} H_n^{-1},$$

we have obtained a C^∞ diffeomorphism f as the limit of $\{f_n\}$ and a homeomorphism H as the limit of H_n . They satisfy

$$f = H R_\alpha H^{-1}.$$

What is left is to show that f is a nonsingular automorphism of (S^1, m) of type III_λ . The purpose of this section is to show that $r(f) \subset \lambda^\mathbb{Z} \cup \{0\}$. For this, it suffices to construct a Borel set Ξ (in fact a Cantor set) of positive measure which satisfies the following proposition.

PROPOSITION 4.1. *If $\xi \in \Xi$ and $f^i(\xi) \in \Xi$ for some $i \in \mathbb{Z}$, then $(f^i)'(\xi) \in \lambda^\mathbb{Z}$.*

Notice that the Radon-Nikodym derivative is just the usual derivative:

$$\frac{d(f^{-i})_* m}{dm}(\xi) = (f^i)'(\xi).$$

For $n \in \mathbb{N}$, let

$$X_n = \bigcap_{j=1}^n (I_j^- \cup I_j^+), \quad Y_n = \bigcap_{j=1}^n (J_j^- \cup J_j^+) \quad \text{and} \quad X = \bigcap_{j=1}^{\infty} (I_j^- \cup I_j^+).$$

Setting

$$\Xi = H(X), \quad \Xi_n = H(X_n),$$

we shall show that Ξ satisfies Proposition 4.1.

Denote $H^{(n+1)} = \lim_{m \rightarrow \infty} h_{n+1} h_{n+2} \cdots h_m$. Thus $H^{(n+1)}$ is a homeomorphism which satisfies

$$H = H_n H^{(n+1)}.$$

For $x \in X_n$, denote by $[x]_n$ the connected component of x in X_n . For $x, x' \in X_n$, denote $x \sim_n x'$ if $[x]_n = [x']_n$.

LEMMA 4.2. *Suppose $n < m$.*

- (1) *If $x, x' \in X_m$ satisfies $x \sim_m x'$, then $x \sim_n y$.*
- (2) *For any $x \in X_n$, $h_{n+1} h_{n+2} \cdots h_m(x) \sim_n x$ and $H^{(n+1)}x \sim_n x$.*

PROOF. (2) is a direct consequence of (3.3). □

First of all, we have to show the following.

LEMMA 4.3. *The set Ξ has positive Lebesgue measure.*

PROOF. Let us compute $m(\Xi_n)$. For $n = 1$, $\Xi_1 = h_1(I_1^- \cup I_1^+)$, and clearly

$$m(\Xi) = m(\hat{h}_1(\hat{I}_1^- \cup \hat{I}_1^+)) = 1 - \lambda'_1.$$

For $n = 2$,

$$\Xi_2 = H((I_1^- \cup I_1^+) \cap (I_2^- \cup I_2^+)) = h_1((I_1^- \cup I_1^+) \cap h_2(I_2^- \cap I_2^+)),$$

by virtue of Lemma 4.2. Also by (3.3), the conditional probabilities of $h_2(I_2^- \cap I_2^+)$ conditioned to I_1^- and I_1^+ coincide with $m(\hat{h}_2(\hat{I}_2^- \cup \hat{I}_2^+))$. On the other hand h_1 is affine on I_1^- and I_1^+ . Therefore

$$m(\Xi_2) = m(\hat{h}_1(\hat{I}_1^- \cup \hat{I}_1^+))m(\hat{h}_2(\hat{I}_2^- \cup \hat{I}_2^+)) = (1 - \lambda'_1)(1 - \lambda'_2).$$

Successive use of (3.3) enables us to conclude that in general for $n \in \mathbb{N}$

$$m(\Xi_n) = \prod_{j=1}^n (1 - \lambda'_j),$$

and thus $m(\Xi)$ is positive by the assumption (3.2). □

REMARK 4.4. The above proof shows that any nonempty open subset of Ξ has positive measure. Furthermore any component of $\Xi_n \cap H(I_n^+)$ has the same measure, as well as any component of $\Xi_n \cap H(I_n^-)$.

LEMMA 4.5. *If $f^i(\xi) \in \Xi$ and $|i| \leq n$, then $f_n^i(\xi) \in H_n(Y_n)$.*

PROOF. Let $x = H^{-1}(\xi)$ and $x_n = H_n^{-1}(\xi) = H^{(n+1)}(x)$. By the assumption, the point $R_\alpha^i(x) = H^{-1}(f^i(\xi))$ belongs to $X \subset X_n$. To show the lemma, it suffices to prove $R_{\alpha_{n+1}}^i(x_n) = H_n^{-1}(f_n^i(\xi))$ is a point of Y_n . This follows once we show that $|R_\alpha^i(x) - R_{\alpha_{n+1}}^i(x_n)|$ is smaller than the width $Q_n^{-1}\delta_n$ of the connected components of $Y_n \setminus X_n$. Now we have

$$|R_\alpha^i(x) - R_{\alpha_{n+1}}^i(x_n)| \leq |i||\alpha - \alpha_{n+1}| + |x_n - x|.$$

For $|i| \leq n$,

$$|i||\alpha - \alpha_{n+1}| \leq n|\alpha - \alpha_{n+1}| < n|\alpha - \alpha_n| < n\delta_n^2 Q_n^{-2} q_n^{-1} < 2^{-1} Q_n^{-1} \delta_n$$

by (3.5), and

$$|x_n - x| < q_{n+1}^{-1} < 2^{-1} Q_n^{-1} \delta_n$$

by (3.4). This completes the proof. \square

LEMMA 4.6. *If ξ , $f^i(\xi) \in \Xi$ and $|i| \leq n$, then $(f_n^i)'(\xi) \in \lambda^{\mathbb{Z}}$.*

PROOF. Let $x = H^{-1}(\xi)$ and $x_k = H_k^{-1}(\xi)$ for any $1 \leq k \leq n$. Then by Lemma 4.2, since $x \in X \subset X_n$, we have $x_k \in X_k$. Therefore $h'_k(x_k) = \lambda^{\pm 1/2}$. Let $y_n = R_{\alpha_{n+1}}(x_n)$ and $y_k = h_{k+1} \cdots h_n(y_n)$. From Lemma 4.5, it follows that $y_n \in Y_n$, and therefore by Lemma 4.2 applied to $\{Y_k\}$, we have $y_k \in Y_k$, and in particular $h'_k(y_k) = \lambda^{\pm 1/2}$. Now we have

$$(f_n^i)'(\xi) = (h_1)'(y_1) \cdots (h_n)'(y_n) \cdot (h_1)'(x_1)^{-1} \cdots (h_n)'(x_n)^{-1} \in \lambda^{\mathbb{Z}}.$$

\square

The proof that Ξ satisfies Proposition 4.1 is complete by the following lemma.

LEMMA 4.7. *If ξ , $f^i(\xi) \in \Xi$, then for any large n , $(f_n)'(\xi) = f'(\xi)$.*

PROOF. Here we shall give a proof which can also be applicable when we show the denseness of type III_∞ diffeomorphisms in later section. Since f_n^i converges to f in C^∞ topology and since $(f_n)'(\xi) - (f_{n+1})'(\xi)$ is either 1 or $\lambda^{\pm 1}$, it must be 1 for any large n . \square

5. Type III_λ : the proof that $\lambda \in r(f)$

Let f and Ξ be as before. Let

$$f_\Xi : \Xi \rightarrow \Xi$$

be the first return map of f . As is well known, the ratio set $r(f_\Xi)$ is the same as $r(f)$. To show that $\lambda \in r(f)$, we need to establish the following proposition.

PROPOSITION 5.1. *For any closed subset $K \subset \Xi$ of positive measure, there exist a point $\xi \in K$ and a number $i \in \mathbb{Z}$ such that $f^i(\xi) \in K$ and $(f^i)'(\xi) = \lambda$.*

In fact this is enough for showing that $\lambda \in r(f_\Xi)$. For, given any Borel subset $A \subset \Xi$, there is a closed subset K of positive measure contained in the set of the points of density of A . If there are ξ and i as in the proposition for this closed subset K , then there is a small interval J_1 centered at ξ of radius r such that $(f^i)' = \lambda$ on $J_1 \cap \Xi \cap f^{-i}(\Xi)$. Consider an interval J_2 centered at $f^i(\xi)$ of radius $(f^i)'(\xi)r$. If r is small enough, then we have

$$m(J_1 \cap A) > (1 - \epsilon)m(J_1) \quad \text{and} \quad m(J_2 \cap A) > (1 - \epsilon)m(J_2),$$

for some small $\epsilon > 0$, since ξ and $f^i(\xi)$ is a point of density of A . Also if r is small, $f^{-1}(J_2)$ almost coincides with J_1 , and f' is almost constant on J_1 . This shows that $B = (J_1 \cap A) \cap f^{-1}(J_2 \cap A)$ has positive measure. For any $\eta \in B$, we have $f^i(\eta) \in A$ and $(f^i)'(\eta) = \lambda$, showing that $\lambda \in r(f_\Xi)$.

The rest of this section is devoted to the proof of Proposition 5.1. It is easier to pass from Ξ to $X = H^{-1}(\Xi)$. So let $\mu = H_*^{-1}m$, and choose once and for all an arbitrary closed subset C of X of positive μ measure. We shall show Proposition 5.1 for $K = H(C)$.

Our overall strategy is as follows. After choosing a large number n , we shall construct points x_k, y_k inductively for $k \geq n$. They satisfy the following conditions, where k is any number bigger than n .

- (1) x_n (resp. y_n) is the midpoint of a component of $X_n \cap I_n^-$ (resp. $X_n \cap I_n^+$), and $x_n \sim_{n-1} y_n$.
- (2) x_k and y_k are the midpoints of components of X_k .
- (3) There is $i \in \mathbb{Z}$, independent of k , such that $y_k = R_{\alpha_k}^i(x_k)$.
- (4) $x_k \sim_{k-1} x_{k-1}$ and $y_k \sim_{k-1} y_{k-1}$.
- (5) $\mu([x_k]_k \cap C) > 0$ and $\mu([y_k]_k \cap C) > 0$.

Let us show that this suffices for our purpose. First of all the two sequences $\{x_k\}$ and $\{y_k\}$ converge by (4). Let

$$x = \lim_{k \rightarrow \infty} x_k, \quad y = \lim_{k \rightarrow \infty} y_k \quad \text{and} \quad \xi = H(x), \quad \eta = H(y).$$

By (5), x and y belong to the closed set C , and hence ξ and η to $K = H(C)$. By (3), we have $R_{\alpha}^i(x) = y$, and hence $f^i(\xi) = \eta$. For any $k \geq 1$ (not just for $k \geq n$), define

$$x'_k = H_k^{-1}(\xi) = H^{(k+1)}(x).$$

By Lemma 4.7, there is m such that $(f_{m-1}^i)'(\xi) = (f^i)'(\xi)$. Notice that f_{m-1}^i can also be written as

$$f_{m-1}^i = H_m R_{\alpha_m}^i H_m^{-1},$$

since $R_{\alpha_m}^i$ commutes with h_m .

Define

$$y'_m = R_{\alpha_m}^i x'_m \quad \text{and} \quad y'_k = h_{k+1} \cdots h_m(y'_m) \quad \text{for } k \leq m.$$

Then we have

$$(f_{m-1}^i)'(\xi) = (h_1)'(y'_1) \cdots (h_m)'(y'_m) \cdot (h_1)'(x'_1)^{-1} \cdots (h_m)'(x'_m)^{-1}.$$

By Lemma 4.2 and (4), we have

$$x'_k \sim_k x_k \quad \text{and} \quad y'_k \sim_k y_k,$$

for $k \geq n$. This shows that

$$h'_n(y'_n)/h'_n(x'_n)^{-1} = \lambda$$

by (1), and for $k > n$

$$(5.1) \quad h'_k(x'_k) = h'_k(y'_k).$$

by (3). On the other hand we have for $k < n$,

$$x'_k \sim_k x_n \sim_k y_n \sim_k y'_k.$$

This shows (5.1) for $k < n$. The proof that $(f^i)'(\xi) = \lambda$ is now complete.

Now we shall construct x_k and y_k for $k \geq n$.

CASE 1 $k = n$: Consider a point of density of C for the measure μ . Then for any $\epsilon > 0$, one can find $n \in \mathbb{N}$ and an interval J bounded by two consecutive fixed points of h_n (a fundamental domain of R_{Q_n}) such that

$$\mu(J \cap C) > (1 - 3^{-1}\epsilon)\mu(J).$$

Then we have

$$J \cap X_n = [x_n]_n \cup [y_n]_n,$$

where x_n (resp. y_n) is the midpoint of $[x_n]_n$ (resp. $[y_n]_n$), and $[x_n]_n \subset I_n^-$ and $[y_n]_n \subset I_n^+$. If n is big enough, then $\mu(J \cap X_n)$ is nearly equal to $\mu(J)$ and one may assume

$$\mu([x_n]_n \cap C) > (1 - 2^{-1}\epsilon)\mu([x_n]_n) \quad \text{and} \quad \mu([y_n]_n \cap C) > (1 - 2^{-1}\epsilon)\mu([y_n]_n).$$

CASE 2 $k = n+1$: Let us call the interval $[jq_{n+1}^{-1}, (j+1)q_{n+1}^{-1}]$ for some $1 \leq j \leq q_{n+1}$ a q_{n+1} -interval. Now $[x_n]_n$ and $[y_n]_n$ are partitioned into q_{n+1} -intervals and one can find a q_{n+1} -interval J_1 (resp. J_2) contained in $[x_n]_n$ (resp. $[y_n]_n$) such that

$$\mu(J_\nu \cap C) > (1 - \epsilon)\mu(J_\nu), \quad \nu = 1, 2.$$

Then there is $1 \leq i \leq q_{n+1}$ such that $R_{\alpha_{n+1}}^i(J_1) = J_2$. Now since

$$\mu(J_\nu \cap X_{n+1}) > (1 - \delta'_{n+1})\mu(J_\nu),$$

and since

$$(1 - \epsilon)(1 - \delta'_{n+1}) > 2/3,$$

where δ'_{n+1} is the constant given by (3.1), either there are more than $2/3$ portion of components $[z]_{n+1}$ among all the components of $J_\nu \cap X_{n+1} \cap I_{n+1}^+$ which satisfies

$$(5.2) \quad m([z]_{n+1} \cap C) \geq (1 - \epsilon)(1 - \delta'_{n+1})m([z]_{n+1}),$$

or else among all the components of $J_\nu \cap X_{n+1} \cap I_{n+1}^-$, for each $\nu = 1, 2$. That is, we can find a component $[x_{n+1}]_{n+1}$ (resp. $[y_{n+1}]_{n+1}$) in J_1 (resp. J_2) such that

$$R_{\alpha_{n+1}}^i(x_{n+1}) = y_{n+1}$$

which satisfies (5.2). Here we have chosen x_{n+1} (resp. y_{n+1}) to be the midpoint of $[x_{n+1}]_{n+1}$ (resp. $[y_{n+1}]_{n+1}$).

CASE 3 *higher k*: Assume we get x_k and y_k which satisfy (2), (3), (4) and

$$(5') \quad \mu([x_k]_k \cap C) > (1 - \epsilon) \prod_{j=1}^k (1 - \delta'_j)^2 \mu([x_k]_k)$$

$$\text{and} \quad \mu([y_k]_k \cap C) > (1 - \epsilon) \prod_{j=1}^k (1 - \delta'_j)^2 \mu([y_k]_k).$$

Since $R_{\alpha_k}^i([x_k]_k) = [y_k]_k$ and since by (3.5)

$$i|\alpha_{k+1} - \alpha_k| < 2q_{n+1}|\alpha_k - \alpha| < 2q_k|\alpha_k - \alpha| < 2\delta_k^2 Q_k^{-2},$$

$R_{\alpha_{k+1}}^i$ maps vast majority of $[x_k]_k$ into $[y_k]_k$.

More precisely the conditional probability of the union of the components of X_{k+1} completely contained in $[x_k]_k \cap R_{\alpha_{k+1}}^{-i}([y_k]_k)$, conditioned to X_{k+1} , is bigger than $(1 - \delta'_{k+1})^2$, and the same is true for $R_{\alpha_{k+1}}^i([x_k]_k \cap [y_k]_k)$. Since

$$(1 - \epsilon) \prod_{j=1}^{k+1} (1 - \delta'_j)^2 > 2/3,$$

just as before, there are $[x_{k+1}]_{k+1}$ in $[x_k]_k$ and $[y_{k+1}]_{k+1}$ in $[y_k]_k$ such that $R_{\alpha_{k+1}}^i(x_{k+1}) = y_{k+1}$ and

$$\mu([x_{k+1}]_{k+1} \cap C) > (1 - \epsilon) \prod_{j=1}^{k+1} (1 - \delta'_j)^2 \mu([x_{k+1}]_{k+1}),$$

$$\mu([y_{k+1}]_{k+1} \cap C) > (1 - \epsilon) \prod_{j=1}^{k+1} (1 - \delta'_j)^2 \mu([y_{k+1}]_{k+1}).$$

This finishes the construction of $\{x_k\}$ and $\{y_k\}$.

6. Type III_∞

We construct \hat{h}_n by almost the same manner as in Section 3. But for n odd we use the slope $\lambda_1^{\pm 1}$ and for n even $\lambda_2^{\pm 1}$, where $1 < \lambda_1 < \lambda_2$. They are chosen so that $\log \lambda_1$ and $\log \lambda_2$ are independent over \mathbb{Q} . We just repeat the argument of Section 5, to show $\lambda_1, \lambda_2 \in r(f)$.

All that need extra care is the validity of the argument which shows that a variant of Proposition 5.1 is sufficient. For this, first of all, we need a variant of Lemma 4.7. It holds true, as is remarked in the proof there. We also need the fact that $(f^i)'$ is locally constant on $\Xi \cap f^{-i}(\Xi)$. To establish this, notice that $\log(f_n^i)'$ converges uniformly to $\log(f^i)'$ and that $(f_n^i)'$ is locally constant on $\Xi \cap f^{-i}(\Xi)$ if $|i| \leq n$ (Compare Lemmata 4.6 and 4.7). Thus for any $\xi \in \Xi \cap f^{-i}(\Xi)$, there is a neighbourhood U of ξ and $n_0 \in \mathbb{N}$ such that if $\eta \in U \cap \Xi \cap f^{-i}(\Xi)$,

- (1) $|\log(f_{n+1}^i)'(\eta) - \log(f_n^i)'(\eta)| < \log \lambda_1$ if $n \geq n_0$ and
- (2) $(f_{n_0}^i)'(\eta) = (f_{n_0}^i)'(\xi)$.

Then we have $(f_n^i)'(\eta) = (f_{n_0}^i)'(\eta)$ for any $n \geq n_0$. The same is true for ξ . This shows $(f^i)'(\eta) = (f^i)'(\xi)$.

7. Type III_0

We construct \hat{h}_n starting at affine functions of slope 3^{-1} and 3^{3^n} , and define the intervals \hat{I}_n^- and \hat{I}_n^+ as in Section 3. Although $m(\hat{I}_n^+) \rightarrow 0$ rapidly as $n \rightarrow \infty$, we have $m(\hat{h}_n(\hat{I}_n^+)) \rightarrow 2/3$ and $m(\hat{h}_n(\hat{I}_n^-)) \rightarrow 1/3$.

First of all let us show that $r(f) \subset \{0, 1\}$. Define X_n, Y_n and Ξ as in Section 4. Let L be a component of $H(X_n)$. We shall show that if n is sufficiently large and $\xi, f^i(\xi) \in L \cap \Xi$, then either $(f^i)'(\xi) = 1$ or $|\log_3(f^i)'(\xi)| > 3^{n-1}$. This is sufficient for our purpose since $m(L \cap \Xi) > 0$.

To show this, notice that there is m bigger than $|i|$ and n such that $f_m^i(\xi) \in H_m(Y_m)$ and $(f_m^i)'(\xi) = (f^i)'(\xi)$. Define $x_j = H_j^{-1}(\xi)$ and $y_j = H_j^{-1}(f_m^i(\xi))$ for $j \leq m$. Then we have

$$\log_3(f_m^i)'(\xi) = \sum_{j=1}^m \log_3 h'_j(y_j) - \sum_{j=1}^m \log_3 h'_j(x_j)$$

Notice that

$$|\log_3 h'_j(y_j) - \log_3 h'_j(x_j)| = 0 \quad \text{or} \quad 3^j - 1, \quad \text{and} \\ \log_3 h'_j(y_j) = \log_3 h'_j(x_j) \quad \text{if} \quad j \leq n,$$

since $\xi, f_m^i(\xi) \in L \cap \Xi$.

Let $k \in [n, m]$ be the largest integer, if any, such that

$$\log_3 h'_k(y_k) \neq \log_3 h'_k(x_k).$$

Then the value

$$|\log_3 h'_k(y_k) - \log_3 h'_k(x_k)|$$

is vastly bigger than

$$\sum_{j=1}^{k-1} |\log_3 h'_j(y_j) - \log_3 h'_j(x_j)|.$$

In fact, computation shows that if n is sufficiently large,

$$|\log_3(f^i)'(\xi)| = |\log_3(f_m^i)'(\xi)| > 3^{n-1},$$

as is required.

What is left is to show that $0 \in r(f)$, since we always have $1 \in r(f)$. To show this, we follow the argument of Section 5 closely and show the following proposition.

PROPOSITION 7.1. *For any closed subset $K \subset \Xi$ of positive measure and any n_0 , there exist a point $\xi \in K$ and a number $i \in \mathbb{Z}$ such that $f^i(\xi) \in K$ and $(f^i)'(\xi) = 3^{3^n-1}$ for some $n > n_0$.*

8. Type II_∞

We begin by constructing \hat{h}_n starting at affine maps of slope $2^{\pm n}$. Define \hat{I}_n^+ just as before. Notice that $m(\hat{I}_n^+) \rightarrow 0$ rapidly as $n \rightarrow \infty$, but that $m(\hat{h}_n(I_n^+)) \rightarrow 1$ rapidly. Define \hat{I}_n^+ as the lift of \hat{I}_n^+ by the Q_n -fold covering map. Denote

$$X^+ = \bigcap_{i=1}^{\infty} I_i^+ \text{ and } \Xi^+ = H(X^+).$$

Then one can show as in Lemma 4.3 that $m(\Xi^+) > 0$. On the other hand it is easy to show that $m(X^+) = 0$. This implies that the unique f -invariant measure H_*m is singular to m . Therefore f cannot be of type II_1 .

On the other hand, we can show the following proposition easily.

PROPOSITION 8.1. *Whenever $\xi, f^i(\xi) \in \Xi^+$ for some $i \in \mathbb{Z}$, then $(f^i)'(\xi) = 1$.*

This completes the proof that $r(f) = \{1\}$. Since f is not of type II_1 , it must be of type II_∞ .

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